

# SOME EXISTENCE THEOREMS IN THE CALCULUS OF VARIATIONS

## IV. ISOPERIMETRIC PROBLEMS IN NON-PARAMETRIC FORM\*

BY

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1. **Heuristic outline of proof.** The preceding notes in this series have already appeared in these Transactions,† and will be cited as I, II, III, respectively.

Here we shall consider isoperimetric problems with one side integral, in the non-parametric form. A heuristic outline of the proof is as follows. We seek to minimize the integral

$$(1.1) \quad \mathcal{J}[y] = \int_{x_0}^x f(x, y, \dot{y}) dx$$

in the class  $K$  of a.c. curves  $y=y(x)$  joining two fixed points  $(x_0, y_0)$  and  $(X, Y)$  and giving a fixed value  $\gamma$  to the integral

$$(1.2) \quad \mathcal{G}[y] = \int_{x_0}^x g(x, y, \dot{y}) dx.$$

Under suitable conditions, we can use as minimizing sequence a sequence of polygons  $\Pi_n: y=y_n(x)$  such that  $\mathcal{J}[y_n]$  tends to its lower bound  $\mu$  on the class  $K$  and  $\gamma_n = \mathcal{G}[y_n]$  tends to  $\gamma$ . Also we may assume that if  $s_n$  is the number of vertices of  $\Pi_n$ , then  $\Pi_n$  is a "best" polygon, in the sense that in the class of polygons of not more than  $s_n$  vertices joining  $(x_0, y_0)$  to  $(X, Y)$  and giving the value  $\gamma_n$  to  $\mathcal{G}[y]$ , the polygon  $\Pi_n$  minimizes  $\mathcal{J}[y]$ .

There are known conditions which must be satisfied by a curve  $\bar{C}: y=\bar{y}(x)$ ,  $(x_0 \leq x \leq X)$ , which is of class  $D'$  and minimizes  $\mathcal{J}(C)$  on the class  $K$ . For our purposes the statement of these conditions is more conveniently written by passing to the parametric notation, by equation (1.1) of I. That is, we write

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† I. *The Dresden corner condition*, these Transactions, vol. 44 (1938), pp. 429–438; II. *Existence theorems for isoperimetric problems in the plane*, *ibid.*, pp. 439–453; III. *Existence theorems for nonregular problems*, these Transactions, vol. 45 (1939), pp. 151–171.

$$(1.3) \quad \begin{aligned} F(z, z') &= F(z^0, z^1, \dots, z^q, z^{0'}, \dots, z^{q'}) \\ &= z^{0'} f(z^0, z^1, \dots, z^q, z^{1'}/z^{0'}, \dots, z^{q'}/z^{0'}) \end{aligned}$$

for  $z^{0'} > 0$ . Correspondingly we write  $\bar{C}$  in the form  $z = \bar{z}(t)$ , ( $0 \leq t \leq 1$ ). Then it is known that there are constants  $\lambda$  and  $\lambda_0$ , not both zero, such that the DuBois-Reymond relation holds:

$$(1.4) \quad H_i(\bar{z}(t), \bar{z}'(t)) = \int_0^t H_{i,t}(\bar{z}(t), \bar{z}'(t)) dt + c_i, \quad i = 0, 1, \dots, q,$$

where  $H = \lambda_0 F - \lambda G$ . If we suppose that  $\bar{C}$  is a normal curve, so that  $\lambda_0 \neq 0$ , we can suppose  $\lambda_0 = 1$ . Let  $\bar{i}$  define a corner of  $\bar{C}$ , and write  $\bar{p} = \bar{z}'(\bar{i}-0)$ ,  $\bar{r} = \bar{z}'(\bar{i}+0)$ . By the Dresden corner condition\*

$$(1.5) \quad \Omega_H(\bar{z}(\bar{i}), \bar{p}, \bar{r}) \leq 0.$$

Also, from (1.4) we derive the Weierstrass-Erdman corner condition:

$$(1.6) \quad H_i(\bar{z}(\bar{i}), \bar{p}) = H_i(\bar{z}(\bar{i}), \bar{r}), \quad i = 0, 1, \dots, q.$$

Of course we cannot expect the polygons  $\Pi_n$  to satisfy these conditions. Nevertheless, each  $\Pi_n$  does minimize  $\mathcal{Y}(C)$  in a certain class of polygons with  $G(C) = \text{const.}$ , and it is therefore not unreasonable to hope that some approximate form of (1.4), (1.5), and (1.6) is satisfied by each  $\Pi_n$ .

Condition (1.6) can be written in a way which does not involve  $\lambda_0$  and  $\lambda$ . We make the following definition:

(1.7) *A set  $A$  of (nonzero) vectors is an approach set at  $z$  if the matrix*

$$\Delta(z, p, r) = \begin{pmatrix} F_i(z, p) - F_i(z, r) \\ G_i(z, p) - G_i(z, r) \end{pmatrix}, \quad i = 0, 1, \dots, q,$$

*has rank less than 2 for each pair  $p, r$  of vectors belonging to  $A$ .*

Then  $\bar{p}$  and  $\bar{r}$  belong to an approach set at  $\bar{z}(\bar{i})$ . For by (1.6) a linear combination of the rows ( $\lambda_0$  times the first row minus  $\lambda$  times the second) vanishes. We shall shortly see that for a fairly large class of functions  $G$  each approach set  $A$  at  $z$  determines a pair of numbers  $\lambda_0 = 1$  and  $\lambda$  for which (1.6) holds.

We now proceed to over-simplify the situation by proceeding as though conditions (1.4), (1.5), and (1.6) actually hold on each  $\Pi_n$ . Furthermore, we assume that at each point  $z$  each approach set contains at most two unit vectors  $p, r$ , and  $\Omega_H(z, p, r) \neq 0$ . We suppose that the sequence  $z_n(t)$  of functions defining the polygons  $\Pi_n$  converges uniformly to a limit function  $z_0(t)$ ;

\* I, Theorem 2.

this is not hard to bring about. Consider a particular value  $t_0$  of  $t$  and a subsequence  $\{\Pi_m\}$  for which  $z'_m(t_0)/|z'_m(t_0)|$  tends to a limit  $p$ . If  $\{t_m\}$  is any sequence of points approaching  $t$ , then by (1.4) the difference

$$H_i(z_m(t_m), z'_m(t_m)) - H_i(z_m(t_0), z'_m(t_0))$$

tends to zero as  $m \rightarrow \infty$ , since it is the integral of a bounded function over an interval which is shrinking to zero. Hence as  $m \rightarrow \infty$  the vector  $z'_m(t_m)/|z'_m(t_m)|$ , if it converges to any limit, converges to a limit  $r$  for which  $H_i(z_0(t_0), r) = H_i(z_0(t_0), p)$ , ( $i=0, 1, \dots, q$ ); that is,  $p$  and  $r$  belong to an approach set at  $z_0(t_0)$ . In other words, if we choose a small arc of  $\Pi_m$  corresponding to  $t$  near  $t_0$ , each side of  $\Pi_m$  has direction either near  $p$  or near  $r$ . To be specific, we assume  $\Omega_H(z_0(t_0), p, r) < 0$ . Then no side of  $\Pi_m$  with direction near  $r$  can be followed by one with direction near  $p$ . For if this happened, at such a corner  $\bar{i}$  we would have  $\Omega_H(z_m(\bar{i}), z'_m(\bar{i}-0), z'_m(\bar{i}+0))$  near  $\Omega_H(z_0(t_0), r, p)$ , which is positive; and this contradicts (1.5). Therefore the arc of  $\Pi_m$  near  $z_0(t_0)$  consists of a succession of sides with directions near  $p$  followed by a succession of sides with directions near  $r$ . That is, the arc consists of two subarcs, each almost linear. This makes it highly plausible that (as would certainly be the case if both subarcs were actually linear) the integrals of  $F$  and  $G$  along this arc of  $\Pi_m$  tend, respectively, to the integrals of  $F$  and  $G$  along the arc of  $z = z_0(t)$  to which they tend. If this could be applied to a succession of subarcs covering the whole range of  $t$ , it would follow that  $\mathcal{F}(\Pi_n)$  and  $\mathcal{G}(\Pi_n)$  tended, respectively, to  $\mathcal{F}(C_0)$  and  $\mathcal{G}(C_0)$ , and  $C_0$  would be the solution sought.

This outline is, of course, over-simplified, and in order to make the proof rigorous it is necessary to depart from it somewhat and add a mass of analytical details. Nevertheless the generalization (8.1) of the assumption that  $\Omega_H(z, p, r) \neq 0$  if  $p, r$  satisfy (1.7) is the vital essence of the proof.

Although we are here concerned with integrals in non-parametric form, in the next paper we shall study integrals in parametric form by closely similar methods. Therefore the lemmas of this paper will be so arranged as to apply simultaneously to both forms of the problem. The chief differences are caused by the requirement  $z^{0'} > 0$  which occurs in studying non-parametric problems. Consequently I shall use the device of enclosing certain statements in square brackets; these are needed if the lemma is understood as applying to non-parametric problems, but should be disregarded for problems in parametric form.

**2. A property of approach sets.** Let us suppose that we are given two integrands  $F(z, z')$ ,  $G(z, z')$  and two distinct fixed points  $z_1, z_2$  in the space of points  $(z^0, z^1, \dots, z^q)$ . We make the following definition:

(2.1)  $K$  is the class of all rectifiable curves  $C$  joining  $z_1$  to  $z_2$ , and  $K[G=l]$  is the subclass of  $K$  consisting of those curves  $C$  of  $K$  for which  $G(C)=l$ .

[In case  $F$  and  $G$  arise from a non-parametric problem, we must restrict our attention to curves representable in the form  $z^i = z^i(z^0)$ , ( $i=1, \dots, q$ ). In this case we define  $K$  as follows:

(2.2)  $K$  is the class of all curves having a.c. representations  $z^i = z^i(z^0)$ , ( $i=1, \dots, q$ ), and joining two fixed points  $z_1, z_2$  with  $z_1^0 < z_2^0$ .  $K[G=l]$  is the subclass of  $K$  consisting of those curves  $C$  for which  $G(C)=l$ .

The points  $z_1, z_2$  will also be denoted by  $(x_0, y_0), (X, Y)$ , respectively.]

The problem considered is that of minimizing  $\mathcal{J}(C)$  on the class  $K[G=l]$ .

[We shall now make a hypothesis in the nature of a continuity requirement on  $F$  and  $G$ .

(2.3) If, as  $n \rightarrow \infty$ , the points  $z_{i,n}, \bar{z}_{i,n}, \zeta_{i,n}$ , and  $\bar{\zeta}_{i,n}$  all converge to a common limit  $z$ , ( $i=0, 1, \dots, q$ ), the numbers  $\theta_{i,n}$  and  $\bar{\theta}_{i,n}$  tend to 0, ( $i=0, 1, \dots, q$ ),  $p_n$  tends to a vector  $p$  with  $p^0 > 0$ , and  $r_n$  tends to a vector  $r$  with  $|r| > 0$ , and if for each  $n$  the matrix

$$(2.4) \quad \begin{pmatrix} F_i(\bar{z}_{i,n}, r_n) - F_i(z_{i,n}, p_n) + \theta_{i,n} \\ G_i(\bar{\zeta}_{i,n}, r_n) - G_i(\zeta_{i,n}, p_n) + \bar{\theta}_{i,n} \end{pmatrix}$$

has rank less than 2, then  $r^0 > 0$ .

Then the following lemma is immediate:

LEMMA 1. If (2.3) holds and the sequences  $z_{i,n}, \bar{z}_{i,n}, \zeta_{i,n}, \bar{\zeta}_{i,n}, \theta_{i,n}, \bar{\theta}_{i,n}, p_n, r_n$  satisfy the conditions of (2.3), then  $\Delta(z, p, r)$  (of (1.7)) has rank less than 2.

For then all the functions in the matrix (2.4) are continuous at  $(z, p, r)$ , and the elements in the matrix tend, respectively, to the elements of the matrix  $\Delta(z, p, r)$ .]

For problems in parametric form the analogue of Lemma 1 is trivial. It is as follows:

LEMMA 1'. If as  $n \rightarrow \infty$  the points  $z_{i,n}, \bar{z}_{i,n}, \zeta_{i,n}$ , and  $\bar{\zeta}_{i,n}$  all converge to a common limit  $z$ , ( $i=0, 1, \dots, q$ ), the numbers  $\theta_{i,n}$  and  $\bar{\theta}_{i,n}$  tend to 0, ( $i=0, 1, \dots, q$ ), and  $p_n$  and  $r_n$  tend to nonzero limit vectors  $p, r$ , and if for each  $n$  the matrix (2.4) has rank less than two, then the matrix  $\Delta(z, p, r)$  has rank less than two.

This is an immediate consequence of the continuity of all the elements of the matrix (2.4).

As is customary in isoperimetric problems, we are often interested in linear combinations of the integrands  $F$  and  $G$ . In order to have a notation

for such combinations we adopt the definition: *For each number  $\lambda$ ,*

$$(2.5) \quad H(z, z'; \lambda) \equiv F(z, z') - \lambda G(z, z').$$

When there is no danger of misunderstanding, we write  $H(z, z')$  in place of  $H(z, z'; \lambda)$ .

Let us make the definition:

(2.6)  $G(z, r)$  is positive quasi-regular normal at  $z_0$  if  $\mathcal{E}_G(z_0, p, r) > 0$  whenever  $|p| > 0$  and  $r \neq kp$ , ( $k \geq 0$ ) [and  $p^0$  and  $r^0$  are positive];  $G(z, r)$  is negative quasi-regular normal at  $z_0$  if  $\mathcal{E}_G(z_0, p, r) < 0$  whenever  $|p| > 0$  and  $r \neq kp$ , ( $k \geq 0$ ) [and  $p^0$  and  $r^0$  are positive];  $G(z, r)$  is quasi-regular normal at  $z_0$  if it is positive or negative quasi-regular normal at  $z_0$ .

Then we state the following interesting useful property of approach sets:

LEMMA 2. *If  $G(z, r)$  is quasi-regular normal at  $z$ , and  $A$  is an approach set at  $z$ , there is a number  $\lambda(z, A)$  such that the function  $H(z, r) \equiv H(z, r; \lambda(z, A)) = F(z, r) - \lambda(z, A)G(z, r)$  satisfies the equations*

$$(2.7) \quad H_i(z, p) = H_i(z, r), \quad i = 0, 1, \dots, q,$$

for all  $p, r$  in  $A$ .

Since  $H(z, r)$  is positively homogeneous of degree zero in  $r$ , it is enough to establish (2.7) for unit vectors in  $A$ . Moreover, it is enough to consider the case in which  $G(z, r)$  is positive quasi-regular normal at  $z$ ; if it is negative quasi-regular normal at  $z$ , we replace it by  $-G(z, r)$ .

Let  $p_1, p_2, p_3$  be any three distinct unit vectors in  $A$ . The relationship

$$(2.8) \quad \begin{aligned} a_1[G_i(z, p_3) - G_i(z, p_2)] + a_2[G_i(z, p_1) - G_i(z, p_3)] \\ + a_3[G_i(z, p_2) - G_i(z, p_1)] = 0, \quad i = 0, 1, \dots, q, \end{aligned}$$

obviously holds if  $a_1 = a_2 = a_3$ . We shall show that it holds for no other  $a_1, a_2, a_3$ . Suppose (2.8) holds. By a cyclic interchange of the subscripts 1, 2, 3, followed, if necessary, by a change of sign of all three, we can bring about the relationship  $a_1 \leq a_2 \leq a_3$ . If we subtract  $a_2$  from each of the numbers  $a_i$ , (2.8) remains valid and assumes the form

$$(2.9) \quad \begin{aligned} (a_1 - a_2)[G_i(z, p_3) - G_i(z, p_2)] \\ + (a_3 - a_2)[G_i(z, p_2) - G_i(z, p_1)] = 0, \quad i = 0, \dots, q. \end{aligned}$$

Now we multiply by  $p_2^i$  and sum over  $i = 0, \dots, q$ ; this yields

$$(2.10) \quad -(a_1 - a_2)\mathcal{E}_G(z, p_3, p_2) + (a_3 - a_2)\mathcal{E}_G(z, p_1, p_2) = 0.$$

But  $-(a_1 - a_2)$  and  $(a_3 - a_2)$  are nonnegative, and their coefficients are positive; so this is only possible if  $a_1 - a_2 = a_3 - a_2 = 0$ .

Given any two distinct unit vectors  $p, r$  of  $A$ , the matrix  $\Delta(z, p, r)$  is (by definition of  $A$ ) of rank 0 or 1. Since its second row is not 0, its first row is a number  $\lambda(p, r)$  times the second. From the form of  $\Delta$  it is clear that  $\lambda(p, r) = \lambda(r, p)$ . We must show that  $\lambda(p, r)$  has the same value for all  $p, r$  in  $A$ . If  $A$  contains only two distinct unit vectors  $p, r$ , this is at once evident from the equation  $\lambda(p, r) = \lambda(r, p)$ . It remains to establish the formula in case  $A$  contains more than two distinct unit vectors. To do this it is sufficient to show (with above notation) that  $\lambda(p_1, p_3) = \lambda(p_1, p_2)$ . For then if we fix  $p_1$  and  $p_2 \neq p_1$ ,  $\lambda(p_1, p_3)$  has the same value for all unit vectors  $p_3$  in  $A$ . That is,  $\lambda(p_1, r)$  is independent of its second argument. Since  $\lambda(p, r) = \lambda(r, p)$ , it is also independent of its first argument, and has only one value.

Suppose then that  $p_1, p_2$ , and  $p_3$  are distinct unit vectors in  $A$ . By definition of  $\lambda(p, r)$  we have

$$F_i(z, p_3) - F_i(z, p_2) = \lambda(p_3, p_2) [G_i(z, p_3) - G_i(z, p_2)],$$

$$F_i(z, p_2) - F_i(z, p_1) = \lambda(p_1, p_2) [G_i(z, p_2) - G_i(z, p_1)],$$

$$F_i(z, p_1) - F_i(z, p_3) = \lambda(p_1, p_3) [G_i(z, p_1) - G_i(z, p_3)].$$

Adding, we obtain (2.8) with  $a_1 = \lambda(p_3, p_2)$ ,  $a_2 = \lambda(p_1, p_3)$ ,  $a_3 = \lambda(p_1, p_2)$ . Hence by the preceding proof we know that  $\lambda(p_1, p_3) = \lambda(p_1, p_2)$ ; so  $\lambda(p, r)$  has but one value, which we denote by  $\lambda(z, A)$ .

Thus (2.7) holds if  $p, r$  are distinct unit vectors in  $A$ . Obviously it holds if  $p = r$ ; and the proof is complete.

Under the hypotheses of Lemma 2, the set  $A$  is an approach set for  $H(z, r)$  at the point  $z$  as defined in III, for equations (2.7) are exactly the equations (1.5) used there to define an approach set. Therefore we have available a geometric interpretation of approach sets. When  $G(z, r)$  is quasi-regular normal at  $z$  and  $A$  is an approach set at  $z$ , then there is a single hyperplane  $u = l_\alpha r^\alpha$  tangent to the hypersurface  $u = F(z, r) - \lambda(z, A)G(z, r)$  at all points  $(u, r)$  with  $r$  in  $A$ . Conversely, let  $\lambda$  be a number, and let  $A$  be the set of (nonzero) vectors  $r$  such that a given hyperplane  $u = l_\alpha r^\alpha$  is tangent to  $u = H(z, r; \lambda)$  whenever  $r$  is in  $A$ . Then as in §7 of III,  $A$  is an approach set for  $H(z, r; \lambda)$ , and equations (2.7) hold. But equations (2.7) imply that the rank of  $\Delta(z, p, r)$  is less than two; so  $A$  is an approach set at  $z$  as defined in (1.7).

This interpretation shows that any approach set  $A$  containing two or more distinct unit vectors has a unique maximal extension, provided that  $G(z, r)$  is quasi-regular normal at  $z$ . For then any two unit vectors of  $A$  determine  $\lambda(z, A)$ , and hence determine  $H(z, r)$ . Let  $u = l_\alpha r^\alpha$  be the hyperplane tangent to  $u = H(z, r)$  whenever  $r$  is in  $A$ . Let  $A^*$  be the set of all nonzero

vectors  $r$  for which  $u = l_\alpha r^\alpha$  is tangent to  $u = H(z, r)$ . Then  $A^*$  is an approach set and contains  $A$ , and is the largest approach set which contains  $A$ .

In view of this lemma it might possibly be more desirable, when  $G$  is not quasi-regular normal at  $z$  to define an approach set at  $z$  to be a set  $A$  such that there exist numbers  $\lambda_0, \lambda$ , not both zero, for which (if we write  $H(z, r) = \lambda_0 F(z, r) - \lambda G(z, r)$ )

$$H_i(z, p) = H_i(z, r), \quad i = 0, 1, \dots, q,$$

for all  $p, r$  in  $A$ . The interest in this remark is rather slight, since the methods of proof which are the central feature of this paper seem not to apply unless  $G(z, r)$  is quasi-regular normal.

**3. Existence of minimizing polygons.** We now add some further hypotheses concerning the integrands. These added requirements are much more stringent for problems in non-parametric form than for those in parametric form.

[(3.1) *For every bounded set  $S_0$  there exist positive numbers  $\delta$  and  $a$  and a number  $b \geq 0$  such that*

$$|F_{\alpha}(z, z')| \leq a |z'| + bF(z_0, z')$$

*wherever  $z_0$  is in  $S_0$  and  $|z - z_0| < \delta$ , and  $z^{0'} > 0$ ; and  $G$  also satisfies this condition.*

(3.2) *On every bounded set  $S_0$  the relations*

$$\lim_{|y'| \rightarrow \infty} f(x, y, y')/|y'| = \infty, \quad \lim_{|y'| \rightarrow \infty} g(x, y, y')/f(x, y, y') = 0$$

*hold uniformly in  $(x, y)$ .*

(3.3) *The points  $z_1 = (x_0, y_0)$  and  $z_2 = (X, Y)$  and the integrals  $\mathcal{F}, G$  have the property that there are constants  $a_0 \geq 0$  and  $a_1$  such that for every number  $H$  there is a bounded set  $S_H$  containing all the a.c. curves  $C: y = y(x)$  joining  $z_1$  and  $z_2$  and having  $a_0 \mathcal{F}(C) + a_1 G(C) < H$ .]*

For problems in parametric form the role of these hypotheses is filled by the following assumption:

(3.4) *The points  $z_1, z_2$  and the integrals  $\mathcal{F}, G$  have the property that if  $K$  is a class of rectifiable curves joining  $z_1$  and  $z_2$  for which  $\mathcal{F}(C)$  is bounded above and  $G(C)$  is bounded, then the curves of  $K$  have uniformly bounded lengths.*

Then we have the following lemma:

LEMMA 3.† *Let  $\nu$  be an integer and  $l$  a number. If the subclass of  $K[G=l]$ , consisting of polygons of not more than  $\nu$  vertices, is not empty, it contains a polygon which minimizes  $\mathcal{F}(C)$  on that class of polygons.*

† For problems in parametric form this is a trivial extension of Lemma 1 of III.

[Let  $y = y_n(x)$ , ( $x_0 \leq x \leq X$ ), be a sequence of polygons of not more than  $\nu$  vertices such that  $G[y_n] = l$  and  $\mathcal{F}[y_n]$  tends to its lower bound  $\mu$ . By Lemma 1 of II, all the curves  $y = y_n(x)$  are in a circle  $Q$ . By Lemma 4 of II, the functions  $y_n(x)$  are equi-absolutely continuous; so by Ascoli's theorem there is a subsequence (which we may suppose to be the original sequence) converging uniformly to a limit function  $y = y_0(x)$ . We see readily that this limit curve is also a polygon of not more than  $\nu$  vertices, and that  $y'_n(x) \rightarrow y'_0(x)$  except at the vertices of  $y = y_0(x)$ . So by Lemma 5 of II we have  $G[y_0] = l$  and  $\mathcal{F}[y_0] \leq \lim \mathcal{F}[y_n] = \mu$ . But by the definition of  $\mu$  we cannot have  $\mathcal{F}[y_0] < \mu$ ; so  $\mathcal{F}[y_0] = \mu$ , and  $y = y_0(x)$  is the polygon sought.]

**4. Choice of a particular minimizing sequence.** Continuing with the hypotheses of the preceding section, we assume that  $l$  is a number such that  $K[G=l]$  is not empty, and we denote by  $\mu$  the greatest lower bound of  $\mathcal{F}(C)$  on the class  $K[G=l]$ . There is [by Lemma 2 of II] a sequence of polygons  $\bar{\Pi}_n$  [of the form  $y = \bar{y}_n(x)$ ] joining  $(x_0, y_0)$  to  $(X, Y)$  and such that  $\lim \mathcal{F}(\bar{\Pi}_n) = \mu$ ,  $\lim G(\bar{\Pi}_n) = l$ . Let  $\nu_n$  be the number of vertices of  $\bar{\Pi}_n$ . By Lemma 3 of the preceding section, there is, for each  $n$ , a polygon  $\Pi_n$  [of the form  $y = y_n(x)$ ] in  $K$  such that

$$(4.1) \quad G(\Pi_n) = G(\bar{\Pi}_n) \rightarrow l,$$

while  $\Pi_n$  minimizes  $\mathcal{F}(C)$  on the class of polygons in  $K$  having not more than  $\nu_n$  vertices and satisfying the equation  $G(C) = G(\bar{\Pi}_n)$ . Hence

$$(4.2) \quad \mathcal{F}(\Pi_n) \leq \mathcal{F}(\bar{\Pi}_n).$$

From this sequence we now select a subsequence  $\bar{\Pi}_{n_i}$  such that

$$(4.3) \quad \lim_{i \rightarrow \infty} \mathcal{F}(\Pi_{n_i}) = \liminf_{n \rightarrow \infty} \mathcal{F}(\Pi_n) \leq \lim_{n \rightarrow \infty} \mathcal{F}(\bar{\Pi}_n) = \mu.$$

We may assume that this subsequence is the whole sequence, so that

$$(4.4) \quad \lim \mathcal{F}(\Pi_n) \text{ exists, } \lim \mathcal{F}(\Pi_n) \leq \mu.$$

By Lemma\* 1 of II the polygons  $\Pi_n$  are all in a circle  $Q$ . [So by Lemma 4 of II the functions  $y_n(x)$  are equi-absolutely continuous, and we can select a subsequence (which we suppose to be the whole sequence) converging uniformly to a limit function  $y_0(x)$ :

$$(4.5) \quad \lim_{n \rightarrow \infty} y_n(x) = y_0(x) \quad \text{uniformly for } x_0 \leq x \leq X.$$

This selection of a subsequence leaves (4.4) still valid.

---

\* Although this lemma was stated for non-parametric problems, its extension to problems in parametric form is trivial.



Hypothesis (3.2) implies that  $f(x, y, y')$  is bounded below for  $(x, y)$  in  $Q$ , say  $f(x, y, y') \geq \kappa$ . Since addition of a constant to  $f$  merely changes  $\mathcal{J}[y]$  by the same number for all curves  $y=y(x)$  of  $K$  and leaves all minima unaffected, we may assume without loss of generality that for all  $(x, y)$  in  $Q$  and all  $y'$

$$(4.6) \quad f(x, y, y') \geq 0.$$

Now we return to the parametric notation.] We have a sequence of polygons  $\{\Pi_n\}$  [converging to a limit curve  $C_0$ ] in  $K$ , and

$$(4.7) \quad \lim G(\Pi_n) = l, \quad \bar{\mu} = \lim \mathcal{J}(\Pi_n) \leq \mu.$$

Also, the lengths of the  $\Pi_n$  are bounded (by (3.4) if the problem is in parametric form, by Lemma 4 of II otherwise), and are not less than  $|z_1 - z_2|$ :

$$(4.8) \quad |z_1 - z_2| \leq \mathcal{L}(\Pi_n) \leq L.$$

On each  $\Pi_n$  we now introduce the parameter  $t=s/\mathcal{L}(\Pi_n)$ , ( $0 \leq s \leq \mathcal{L}(\Pi_n)$ ). Thus  $\Pi_n$  has the representation

$$\Pi_n: \quad z = z_n(t), \quad 0 \leq t \leq 1,$$

with

$$(4.9) \quad 0 < |z_1 - z_2| \leq |\dot{z}_n(t)| \leq \mathcal{L}(\Pi_n) \leq L$$

except at vertices of  $\Pi_n$ .

Since the functions  $z_n$  all satisfy the same Lipschitz condition, there is a subsequence (we suppose it the whole sequence) which converges uniformly to a limit function  $z_0(t)$  on  $[0, 1]$ :

$$(4.10) \quad \lim_{n \rightarrow \infty} z_n(t) = z_0(t) \quad \text{uniformly on } [0, 1].$$

The curve  $z=z_0(t)$  is therefore a limit curve\* of the  $\Pi_n$ . [But these have the unique limit  $C_0$ ; so  $z=z_0(t)$ , ( $0 \leq t \leq 1$ ), is a representation of  $C_0$ .]

Now we define

$$(4.11) \quad \phi_n(t) = \int_0^t F(z_n(t), \dot{z}_n(t)) dt, \quad n = 0, 1, \dots$$

[By (4.6), these functions are monotonic increasing; and they are bounded, for

$$0 = \phi_n(0) \leq \phi_n(t) \leq \phi_n(1) = \int_0^1 F(z_n, z_n') dt \rightarrow \bar{\mu} \leq \mu,$$

---

\* If the reader is omitting bracketed statements, we here define  $C_0$  to be the curve  $z=z_0(t)$ , ( $0 \leq t \leq 1$ ).

by (4.7).] So [by Helly's theorem] we can select a subsequence of  $\{\phi_n\}$  (we suppose it the whole sequence) which converges\* for all  $t$  in  $[0, 1]$  to a limit function  $\phi(t)$ :

$$(4.12) \quad \lim_{n \rightarrow \infty} \phi_n(t) = \phi(t), \quad 0 \leq t \leq 1.$$

For this subsequence, (4.7) and (4.10) remain valid. By Lemma 3 of II, the functions

$$(4.13) \quad \gamma_n(t) \equiv \int_0^t G(z_n(t), \dot{z}_n(t)) dt, \quad 0 \leq t \leq 1, \quad n = 0, 1, \dots,$$

are equi-absolutely continuous. Hence we can select a subsequence (we suppose it to be the whole sequence) which converges uniformly on  $[0, 1]$  to a limit function  $\gamma(t)$ :

$$(4.14) \quad \lim_{n \rightarrow \infty} \gamma_n(t) = \gamma(t) \quad \text{uniformly on } [0, 1].$$

For this subsequence (4.7), (4.10), and (4.12) remain valid.

Our task is now to prove

$$(4.15) \quad \gamma(1) = \gamma_0(1), \quad \phi(1) \geq \phi_0(1).$$

For these are the same as

$$(4.16) \quad \lim G(\Pi_n) = G(C_0), \quad \lim \mathcal{J}(\Pi_n) \geq \mathcal{J}(C_0).$$

If we can establish them, then the first equation shows that  $C_0$  is in  $K[G=l]$ , by (4.1). Hence  $\mathcal{J}(C_0) \geq \mu$ . But the second part of (4.16), with (4.7), gives  $\mathcal{J}(C_0) \leq \mu$ ; hence  $\mathcal{J}(C_0) = \mu$ , and  $y = y_0(x)$  is the curve sought.

It is worth noticing that all the properties of the sequence which have been established here are also possessed by every subsequence of this sequence.

**5. Convergence of directions at one point.** We observe that,  $\phi(t)$ ,  $\phi_0(t)$ , [being monotonic increasing, and]  $\dot{\gamma}(t)$ ,  $\dot{\gamma}_0(t)$ , being a.c., there is a set  $E$  of measure 1 contained in the open interval  $0 < t < 1$  and having the following property:

(5.1) *The derivatives  $\phi'(t)$ ,  $\phi'_0(t)$ ,  $\gamma'(t)$ , and  $\gamma'_0(t)$  exist and are finite for all  $t$  in  $E$ .*

We now wish to prove the following lemma:

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\* For problems in parametric form we note that the integrands in (4.11) are uniformly bounded. So the  $\phi_n(t)$  all satisfy the same Lipschitz condition, and a subsequence can be selected which converges uniformly to a limit  $\phi_0(t)$ . A like remark applies to (4.14).

LEMMA 4. If  $t_1$  belongs to  $E$ , there is a subsequence  $\{\Pi_m\}$ , ( $m = n_1, n_2, \dots$ ), of  $\{\Pi_n\}$  and a sequence  $\{t_m\}$  of points of  $[0, 1]$  such that  $\lim_{m \rightarrow \infty} t_m = t_1$ , and  $z'_m(t_m)$  converges\* to a limit  $p$  [and  $p^0 > 0$ ].

[Let  $\epsilon_i$  be a sequence of positive numbers less than  $1 - t_1$ . The numbers  $[\phi(t_1 + \epsilon_i) - \phi(t_1)]/\epsilon_i$  tend to  $\phi'(t_1)$ , which is finite. Hence there is an  $h$  such that

$$|\phi(t_1 + \epsilon_i) - \phi(t_1)|/\epsilon_i < h.$$

Therefore, by (4.14), the inequality

$$(5.2) \quad |\phi_n(t_1 + \epsilon_i) - \phi_n(t_1)|/\epsilon_i < h + 1$$

holds for all  $n$  greater than a certain  $n(i)$ . We choose a sequence  $\{n_i\}$  such that  $n_1 < n_2 < \dots$  and  $n_i > n(i)$ . Since  $\phi_n$  is a.c. and (by 4.6) monotonic increasing, from (5.2) we see that there is a set of positive measure in  $[t_1, t_1 + \epsilon_i]$  on which  $|\phi'_{n_i}(t)| < h + 1$ . In this set we choose a  $t_{n_i}$  which does not define a vertex of  $\Pi_{n_i}$ ; and we use the letter  $m$  to replace  $n_i$ . Then  $z'_m(t_m)$  is defined, and

$$F(z_m(t_m), z'_m(t_m)) = \phi'_m(t_m) < h + 1.$$

Therefore by (4.9)

$$(5.3) \quad F(z_m(t_m), z'_m(t_m)/|z'_m(t_m)|) < (h + 1)/|z'_m(t_m)| \leq (h + 1)/|z_1 - z_2|.$$

Hypothesis (3.2), in its parametric form, states that there is a  $\delta > 0$  such that

$$(5.4) \quad F(z, z') > \frac{h + 1}{|z_1 - z_2|}, \quad z \text{ in } Q, \quad |z'| = 1, \quad z^{0'} < \delta.$$

Therefore from (5.3) we conclude

$$(5.5) \quad z_m^{0'}(t_m) \geq \delta.$$

From the bounded sequence  $\{z'_m(t_m)\}$  we now choose a subsequence converging to a limit  $p$ ; we may suppose this subsequence to be the whole sequence. Then, by (5.5),  $p^0 \geq \delta$ , and the lemma is established.]

**6. The directions of the sides of the polygons.** In our sketch of the proof, in the introduction, the two essentials were first to show that the directions of the sides of the  $\Pi_n$  were, on short arcs, near approach sets, and second to establish an order relation between the sides. The next lemma treats the first of these two needs.

\* For problems in parametric form this is an immediate consequence of the Bolzano-Weierstrass theorem, since the lengths  $|z'_m(t_m)|$  are bounded.

LEMMA 5. Let  $\{\Pi_n\}$  be the sequence of polygons specified in §4, and let  $t_0$  be a point of  $E$ . Then either

(i) there is a subsequence  $\{\Pi_m\}$  of  $\{\Pi_n\}$  and a sequence of numbers  $t_m \rightarrow t_0$  such that each  $t_m$  is contained in an interval  $h_m \leq t_m \leq k_m$  on which  $z_m(t)$  is linear, and  $k_m - h_m$  does not tend to 0, or

(ii) there is an approach set  $A$  at  $z_0(t_0)$  and a subsequence  $\{\Pi_m\}$  such that if  $t_m \rightarrow t_0$  and  $z_m(t_m)$  is not a vertex\* of  $\Pi_m$ , then all accumulation points of the  $z'_m(t_m)$  belong to  $A$ .

Suppose that (i) does not hold. Let  $\{t_n\}$  be a sequence tending to  $t_0$ , and let  $h_n, k_n$  define the initial and final points of the side of  $\Pi_n$  to which  $t_n$  belongs. Then  $h_n \rightarrow t_0$  and  $k_n \rightarrow t_0$ . For otherwise either  $t_n - h_n$  or  $k_n - t_n$  would not tend to 0, and  $z_n(t)$  would be linear on the intervals  $(h_n, t_n)$  and  $(t_n, k_n)$ .

By Lemma 4, there is a subsequence  $\{\Pi_h\}$  of  $\{\Pi_n\}$  and a sequence of numbers  $t_h^1$  such that  $t_h^1 \rightarrow t_0$  and  $z'_h(t_h^1)$  tends to a limit  $p_1$  [with  $p_1^0 > 0$ ]. If for every sequence  $t_h \rightarrow t_0$  the relation  $z'_h(t_h) \rightarrow p_1$  holds, then case (ii) holds. Otherwise, for some sequence  $t_h^2 \rightarrow t_0$  the vectors  $z'_h(t_h^2)$  have an accumulation point  $p_2 \neq p_1$ . For a subsequence  $\{t_w^2\}$  we have  $z'_w(t_w^2) \rightarrow p_2$ , while it is still true that  $z'_w(t_w^1) \rightarrow p_1$ .

To be specific, we suppose  $t_w^1 < t_w^2$  for all  $w$ . Let  $v_0, v_1, \dots, v_{s+1}$  be values of  $t$  which define successive vertices of  $\Pi_w$  and satisfy  $v_0 < t_w^1 < v_1 < \dots < v_s < t_w^2 < v_{s+1}$ . By the first paragraph,  $v_0$  and  $v_{s+1}$  tend to  $t_0$  as  $w \rightarrow \infty$ . If we displace the vertices  $z_w(v_1), \dots, z_w(v_s)$  all by the same amount  $(0, 0, \dots, 0, \tau, 0, \dots, 0)$ , where  $\tau$  is in the  $(j+1)$ st place, we obtain a polygon  $\Pi_w^j(\tau)$ . [For small  $\tau$  this can be represented in the form  $z^i = z^i(z^0)$ ,  $(i=1, \dots, n)$ .] From formula (2.8) of I we have

$$\begin{aligned}
 \left. \frac{d}{d\tau} \mathcal{F}(\Pi_w^j(\tau)) \right|_{\tau=0} &= F_j(z_w(t), z'_w(t)) - F_j(z_w(\bar{t}), z'_w(\bar{t})) \\
 &+ \int_{v_0}^{v_1} \frac{t - v_0}{v_1 - v_0} F_{zi}(z_w, z'_w) dt \\
 &+ \int_{v_s}^{v_{s+1}} \frac{t - v_{s+1}}{v_s - v_{s+1}} F_{zi}(z_w, z'_w) dt \\
 &+ \int_{v_1}^{v_s} F_{zi}(z_w, z'_w) dt,
 \end{aligned}
 \tag{6.1}$$

where  $v_0 < \bar{t} < v_1$  and  $v_s < \bar{t} < v_{s+1}$ .

Let us denote the sum of the last three terms in (5.6) by  $\theta_j$ . [By (3.1) and (4.9) the inequality

\* This requirement is not essential.

$$(6.2) \quad \begin{aligned} |\theta_j| &\leq \int_{v_0}^{v_{s+1}} \{a|\dot{z}_w| + bF(z_w, \dot{z}_w)\} dt \\ &\leq aL|v_{s+1} - v_0| + b\{\phi_w(v_{s+1}) - \phi_w(v_0)\} \end{aligned}$$

holds.\*] Also, on the intervals  $(v_0, v_1)$  and  $(v_s, v_{s+1})$  the derivative  $z'_w(t)$  is constant. Hence (5.1) can be written

$$(6.3) \quad D_j \equiv \frac{d}{dt} \mathcal{F}(\Pi_w^j(\tau)) \Big|_{\tau=0} = F_j(z_w(\bar{t}), z'_w(t_w^2)) - F_j(z_w(\bar{t}), z'_w(t_w^1)) + \theta_j.$$

A similar result holds for  $G(C)$ :

$$(6.4) \quad \bar{D}_j \equiv \frac{d}{d\tau} \mathcal{G}(\Pi_w^j(\tau)) \Big|_{\tau=0} = G_j(z_w(\bar{t}), z'_w(t_w^2)) - G_j(z_w(\bar{t}), z'_w(t_w^1)) + \bar{\theta}_j,$$

$$v_0 < \bar{t} < v_1, \quad v_s < \bar{t} < v_{s+1},$$

where for non-parametric problems

$$(6.5) \quad |\bar{\theta}_j| \leq \bar{a}L(v_{s+1} - v_0) + \bar{b}\{\gamma_w(v_{s+1}) - \gamma_w(v_0)\},$$

while for parametric problems

$$(6.5') \quad |\bar{\theta}_j| \leq \bar{M}(v_{s+1} - v_0).$$

Now we show that the matrix

$$(6.6) \quad \begin{pmatrix} D_0 & D_1 & \cdots & D_q \\ \bar{D}_0 & \bar{D}_1 & \cdots & \bar{D}_q \end{pmatrix}$$

is of rank less than 2. Otherwise, suppose, say, that the determinant

$$(6.7) \quad \begin{vmatrix} D_j & D_k \\ \bar{D}_j & \bar{D}_k \end{vmatrix}$$

is not 0. We form the polygon  $\Pi_w(\tau, \sigma)$  by displacing each of the vertices  $z_w(v_1), \dots, z_w(v_s)$  by the amount  $(0, 0, \dots, \tau, 0, \dots, 0, \sigma, \dots, 0)$ , where  $\tau$  is in the  $(j+1)$ st place and  $\sigma$  in the  $(k+1)$ st. Then the jacobian of the functions  $\mathcal{F}(\Pi_w(\tau, \sigma))$ ,  $\mathcal{G}(\Pi_w(\tau, \sigma))$  at  $\tau=\sigma=0$  is exactly the determinant (6.7). The equations

$$(6.8) \quad \begin{aligned} \mathcal{F}(\Pi_w(\tau, \sigma)) + u - \mathcal{F}(\Pi_w) &= 0, \\ \mathcal{G}(\Pi_w(\tau, \sigma)) - \mathcal{G}(\Pi_w) &= 0 \end{aligned}$$

have the initial solutions  $\tau=\sigma=u=0$ . By the implicit functions theorem, we can solve for  $\tau, \sigma$  as functions of  $u$ , defined for all  $u$  near 0. However, if  $u > 0$

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\* For parametric problems this is replaced by the inequality (6.2')  $|\theta_j| \leq M(v_{s+1} - v_0)$ , where  $M$  is an upper bound for  $|F_j(z, z')|$  for  $z$  near  $C_0$  and  $|z'| \leq L$ .

these functions would define values of  $\tau, \sigma$  for which (6.8) held; that is,

$$(6.9) \quad \begin{aligned} G(\Pi_w(\tau, \sigma)) &= G(\Pi_w), \\ \mathcal{J}(\Pi_w(\tau, \sigma)) &< \mathcal{J}(\Pi_w). \end{aligned}$$

This is incompatible with the minimizing property of the polygons  $\Pi_w$ . Hence the matrix (6.6) is of rank less than two.

We now write (6.6) in the form

$$(6.10) \quad \begin{pmatrix} F_j(z_w(\bar{t}_w), z_w'(t_w^2)) - F_j(z_w(\bar{t}_w), z_w'(t_w^1)) + \theta_j \\ G_j(z_w(\bar{t}_w), z_w'(t_w^2)) - G_j(z_w(\bar{t}_w), z_w'(t_w^1)) + \bar{\theta}_j \end{pmatrix}.$$

For parametric problems inequalities (6.2') and (6.5') make it evident that  $\theta_j$  and  $\bar{\theta}_j$  approach 0 with  $v_{s+1} - v_0$ . But  $v_{s+1}$  and  $v_0$  approach  $t_0$  as  $w \rightarrow \infty$ ; so  $\theta_j$  and  $\bar{\theta}_j$  approach 0 as  $w \rightarrow \infty$ . [For problems in non-parametric form we use (6.2) and (6.5). Since  $\phi'(t_0)$  exists, and  $\phi(t)$  is monotonic increasing, for every positive  $\epsilon$  there is a  $\delta > 0$  such that

$$(6.11) \quad \phi(t_0) - \epsilon < \phi(t_0 - \delta) \leq \phi(t_0) \leq \phi(t_0 + \delta) < \phi(t_0) + \epsilon;$$

and if  $w$  is large enough,

$$|\phi_w(t_0 - \delta) - \phi(t_0 - \delta)| < \epsilon, \quad |\phi_w(t_0 + \delta) - \phi(t_0 + \delta)| < \epsilon.$$

Hence if

$$(6.12) \quad t_0 - \delta < t < t_0 + \delta,$$

then

$$(6.13) \quad \phi(t_0) - 2\epsilon < \phi_w(t_0 - \delta) \leq \phi_w(t) \leq \phi_w(t_0 + \delta) < \phi(t_0) + 2\epsilon.$$

For all large  $w$  the points  $v_0$  and  $v_{s+1}$  both satisfy (6.12), since they both approach  $t_0$  as  $w \rightarrow \infty$ ; so by (6.13) we have

$$(6.14) \quad |\phi_w(v_{s+1}) - \phi_w(v_0)| < 4\epsilon.$$

Thus  $\phi_w(v_{s+1}) - \phi_w(v_0)$  approaches 0 as  $w \rightarrow \infty$ . The other term  $aL(v_{s+1} - v_0)$  also tends to 0, as previously observed; so  $\theta_j$  tends to zero with  $1/w$ . The proof that  $\bar{\theta}_j \rightarrow 0$  is easier. The functions  $\gamma_w(t)$  are equi-absolutely continuous; so when  $w \rightarrow \infty$  (and consequently  $v_{s+1} - v_0 \rightarrow 0$ ) the term  $\gamma_w(v_{s+1}) - \gamma_w(v_0)$  tends to 0. The other term in (5.10) also tends to 0; hence  $\bar{\theta}_j$  approaches 0 as  $w \rightarrow \infty$ .]

Since  $z_w(t)$  tends uniformly to  $z_0(t)$ , and the numbers  $\bar{t}_w, \bar{\bar{t}}_w, \bar{t}_w$ , and  $\bar{t}_w$  all tend to  $t_0$ , we have

$$(6.15) \quad \lim z_w(\bar{t}_w) = \lim z_w(\bar{\bar{t}}_w) = \lim z_w(\bar{t}_w) = \lim z_w(\bar{t}_w) = z_0(t_0).$$

Finally, we already know that  $\lim z_w'(t_w^1) = p_1$  [where  $p_1^0 > 0$ ], and that

$\lim z'_w(t_w^2) = p_2$ . Therefore [from (2.4) it follows that  $p_2^0 > 0$ , and by Lemma 1] (for parametric problems by Lemma 1') the vectors  $p_1$  and  $p_2$  belong to an approach set at  $z_0(t_0)$ , and the matrix

$$(6.16) \quad \begin{pmatrix} F_j(z_0(t_0), p_h) & F_j(z_0(t_0), p_k) \\ G_j(z_0(t_0), p_h) & G_j(z_0(t_0), p_k) \end{pmatrix}$$

has rank less than two, where  $h=1$  and  $k=2$ .

Now we can define the approach set  $A$  to be the maximal approach set at  $z$  containing  $p_1$  and  $p_2$  (see remark at end of §2).

Suppose now that there is a third sequence  $t_w^3 \rightarrow t_0$  such that  $z'_w(t_w^3)$  has an accumulation point  $p_3$  different from  $p_1$  and  $p_2$ . We select a subsequence  $\{l\}$  of the integers  $\{w\}$  such that  $z'_l(t_l^3) \rightarrow p_3$ . By the preceding argument, the matrix (6.16) has rank less than two if  $h=1$  and  $k=3$  or if  $h=2$  and  $k=3$ . So  $p_3$  belongs to an approach set at  $z$  containing  $p_1$  and  $p_2$ , and therefore belongs to the maximal set  $A$ . The lemma is thus established.

**7. A lemma on the order of sides.** The next lemma is the one by which the  $\Omega$ -function is made useful to us.

**LEMMA 6.** *Let  $p^*, r^*$  be distinct unit vectors [with  $p^{*0} > 0$  and  $r^{*0} > 0$ ] belonging to an approach set  $A$  at  $z^*$ . Let the inequality*

$$\Omega_H(z^*, p^*, r^*) < 0$$

*hold, where  $H(z, z') = H(z, z'; \lambda(z^*, A)) = F(z, z') - \lambda(z^*, A)G(z, z')$ . Let  $G(z, r)$  be quasi-regular normal (either positive or negative) at  $z^*$ . Then there is a  $\delta > 0$  such that if  $ABC$  is a polygon of two sides lying entirely in the  $\delta$ -neighborhood of  $z^*$ , the side  $AB$  having direction  $r$  such that  $|r - r^*| < \delta$  and the side  $BC$  having direction  $p$  such that  $|p - p^*| < \delta$ , then  $ABC$  does not minimize  $\mathcal{F}(C)$  in the class of all two-sided polygons  $ADC$  joining  $A$  and  $C$  and having  $G(ADC) = G(ABC)$ .*

To be specific we assume that  $G(z, r)$  is positive quasi-regular normal at  $z^*$ . This involves no loss of generality. For if  $G(z, r)$  is negative quasi-regular normal at  $z^*$  we have only to replace it by  $-G(z, r)$ . This causes  $\lambda(z^*, A)$  to be replaced by  $-\lambda(z^*, A)$ , and  $H(z, r)$  is unchanged.

In order to avoid repetition we assume that every positive constant introduced in this proof is less than one, and we let  $M$  be an upper bound for the sums

$$2 \sum_{i=0}^q \{ |H_{zi}(z, r)| + |G_{zi}(z, r)| \}$$

for all  $(z, r)$  such that  $|z - z^*| \leq 5$  and  $|r| = 1$  [and  $r^0 \geq \frac{1}{2} \min(p^{*0}, r^{*0})$ ]. We shall connote that a vector is a unit vector by giving it a subscript  $u$ .

By hypothesis,  $\Omega_H(z^*, p^*, r^*) < -3m < 0$ . Thus by continuity there are positive numbers  $\zeta, \delta$  (we take  $\zeta, \delta, m$  all less than 1) such that

$$(7.1) \quad \Omega_H(z, p_u, r_u) < -2m, \quad [p_u^0 > \frac{1}{2} \min(p^{*0}, r^{*0}), \quad r_u^0 > \frac{1}{2} \min(p^{*0}, r^{*0})]$$

if  $|z - z^*| < 5\zeta$ ,  $|p_u - p^*| < 3\delta$ , and  $|r_u - r^*| < 3\delta$ .

Again by hypothesis, the numbers  $\mathcal{E}_G(z^*, p^*, r^*)$  and  $\mathcal{E}_G(z^*, r^*, p^*)$  are positive. Let  $3e$ , ( $0 < e < 1$ ), be smaller than the smaller of them. Then if  $\zeta$  and  $\delta$  are small enough,

$$(7.2) \quad G(z, r_u) - r_u^\alpha G_\alpha(\bar{z}, p_u) > 2e, \quad G(z, p_u) - p_u^\alpha G_\alpha(\bar{z}, r_u) > 2e$$

if  $z$  and  $\bar{z}$  are in the  $5\zeta$ -neighborhood of  $z^*$  and  $|p_u - p^*| < 3\delta$ ,  $|r_u - r^*| < 3\delta$ . By definition of  $M$ ,

$$(7.3) \quad |\Omega_G(z, p_u, r_u)| < M \text{ if } |z - z^*| < 5\zeta, \quad |p_u - p^*| < 3\delta, \quad |r_u - r^*| < 3\delta.$$

By Lemma 2,  $H_i(z^*, p^*) = H_i(z^*, r^*)$ , ( $i=0, 1, \dots, q$ ). If we multiply by  $p^{*i}$  or by  $r^{*i}$  and sum, we find

$$(7.4) \quad \mathcal{E}_H(z^*, p^*, r^*) = \mathcal{E}_H(z^*, r^*, p^*) = 0.$$

Then if  $\zeta$  and  $\delta$  are small enough, we find, as in establishing (7.2), that

$$(7.5) \quad \begin{aligned} |H(z, r_u) - r_u^\alpha H_\alpha(\bar{z}, p_u)| &< em/M, \\ |H(z, p_u) - p_u^\alpha H_\alpha(\bar{z}, r_u)| &< em/M \end{aligned}$$

if  $|z - z^*| < 5\zeta$ ,  $|\bar{z} - z^*| < 5\zeta$ ,  $|p_u - p^*| < 3\delta$ , and  $|r_u - r^*| < 3\delta$ .

Let  $ABC$  be a two-sided polygon lying in the  $\zeta$ -neighborhood of  $z^*$ . We denote  $A, B, C$  by  $z_1, z_2, z_3$ , respectively, and for their directions and lengths we use the symbols

$$(7.6) \quad \begin{aligned} p_1 &= (z_2 - z_1)/|z_2 - z_1|, & p_2 &= (z_3 - z_2)/|z_3 - z_2|, \\ l_1 &= |z_2 - z_1|, & l_2 &= |z_3 - z_2|; \end{aligned}$$

and we assume

$$(7.7) \quad |p_1 - r^*| < \delta, \quad |p_2 - p^*| < \delta.$$

We now distinguish the two cases  $l_1 \geq l_2$ ,  $l_1 < l_2$ . In the former case we write

$$(7.8) \quad z_\tau = z_1 + (1 + \tau)(z_3 - z_2);$$

in the latter case,

$$(7.9) \quad z_\tau = z_3 + (1 + \tau)(z_1 - z_2).$$

The two cases differ only trivially; so we suppose, to be specific, that  $l_1 \geq l_2$ ; the alterations needed to cover the other case are obvious. If  $|\tau| < 1$ , then  $|z_\tau - z_1| < 2|z_3 - z_2| = 2l_2 < 4\zeta$ ; so  $z_\tau$  is in the  $5\zeta$ -neighborhood of  $z^*$ . Therefore



for  $|\tau| < 1$  the quadrilateral  $z_1 z_2 z_3 z_\tau$  lies entirely in the sphere  $|z - z^*| < 5\zeta$ .

By an elementary trigonometric computation we find that if  $|\tau| < \delta l_1/l_2$  (and therefore if  $|\tau| < \delta$ , since  $l_1 \geq l_2$ ), then the unit vector

$$(7.10) \quad p(\tau) \equiv (z_3 - z_\tau) / |z_3 - z_\tau|$$

satisfies the inequality  $|p(\tau) - p_1| < 2\delta$ . Hence, by (7.7),

$$(7.11) \quad |p(\tau) - r^*| < 3\delta \quad \text{if} \quad |\tau| < \delta.$$

Also, if  $|\tau| < 1$ , then (by (7.8))

$$(7.12) \quad |z_\tau - z_3| = |z_1 - z_2 + \tau(z_3 - z_2)| \leq |z_1 - z_2| + |z_3 - z_2| \leq 2l_1.$$

We henceforth assume

$$(7.13) \quad |\tau| < \delta.$$

The point  $z_\tau$  will also be denoted by  $D_\tau$ . The figure  $ABCD_0$  is a parallelogram, by (7.8). Hence by formula (3.19) of I

$$(7.14) \quad \mathcal{K}(AD_0C) - \mathcal{K}(ABC) = -l_1 l_2 \Omega_H(\bar{z}, p_1, p_2),$$

$$(7.15) \quad \mathcal{G}(AD_0C) - \mathcal{G}(ABC) = -l_1 l_2 \Omega_G(\bar{z}, p_1, p_2),$$

where  $\bar{z}$  and  $\tilde{z}$  are in the parallelogram  $ABCD_0$ . If we recall that  $\Omega_H(z, p, r) = -\Omega_H(z, r, p)$ , these inequalities, with (7.7), (7.1), and (7.3), yield

$$(7.16) \quad \mathcal{K}(AD_0C) - \mathcal{K}(ABC) < -2l_1 l_2 m,$$

$$(7.17) \quad |\mathcal{G}(AD_0C) - \mathcal{G}(ABC)| < l_1 l_2 M.$$

Now we compute the derivative of  $\mathcal{G}(AD_\tau C)$  with respect to  $\tau$ . The side  $AD$  is defined by the equation  $z = z_1 + t(z_3 - z_2)$ , ( $0 \leq t \leq 1 + \tau$ ). So

$$(7.18) \quad \begin{aligned} \frac{d}{d\tau} \mathcal{G}(AD_\tau) &= \frac{d}{d\tau} \int_0^{1+\tau} G(z_1 + t(z_3 - z_2), z_3 - z_2) dt \\ &= G(z_\tau, z_3 - z_2) = l_2 G(z_\tau, p_2). \end{aligned}$$

The derivative of  $G(D_\tau C)$  we compute by formula (2.8) of I, taking  $\pi_1 = z_3 - z_2$ ,  $\pi_2 = 0$ . The side  $D_\tau C$  is defined by the equation  $z = (1-t)z_\tau + tz_3$ , ( $0 \leq t \leq 1$ ); so

$$(7.19) \quad \begin{aligned} \frac{d}{d\tau} \mathcal{G}(D_\tau C) &= -G_\alpha(\bar{z}, z_3 - z_2) [z_3^\alpha - z_2^\alpha] \\ &\quad + \int_0^1 G_{z^\alpha}((1-t)z_\tau + tz_3, z_3 - z_\tau) [z_3^\alpha - z_2^\alpha] (1-t) dt \\ &= -l_2 G_\alpha(\bar{z}, p_2) p_2^\alpha \\ &\quad + \int_0^1 G_{z^\alpha}((1-t)z_\tau + tz_3, p(\tau)) p_2^\alpha l_2 |z_3 - z_\tau| (1-t) dt, \end{aligned}$$

where  $\bar{z}$  is on the line segment  $D, C$ . By the definition of  $M$ , together with (7.11) and (7.12), the integrand in (7.19) does not exceed  $2Ml_1l_2(1-t)$  in absolute value. Hence the absolute value of the integral is at most  $Ml_1l_2$ , and if we add (7.18) and (7.19) we find

$$(7.20) \quad \frac{d}{d\tau} G(AD, C) = l_2[G_\alpha(z_\tau, p_2) - p_2^\alpha G_\alpha(\bar{z}, p(\tau))] + \theta Ml_1l_2, \quad |\theta| \leq 1.$$

The points  $z_\tau$  and  $\bar{z}$  are in the  $5\zeta$ -neighborhood of  $z^*$ , and (7.11) and (7.7) hold; so by (7.2),

$$(7.21) \quad \frac{d}{d\tau} G(AD, C) > 2l_2e + \theta Ml_1l_2, \quad |\theta| \leq 1,$$

if  $|\tau| \leq \delta$ .

The conditions thus far imposed on  $\zeta$  and  $\delta$  merely require that they be sufficiently small. We now require

$$(7.22) \quad \zeta < e\delta/2M^2.$$

Then

$$(7.23) \quad l_1 < 2\zeta < e\delta m/M^2 < e/M,$$

and (7.21) yields

$$(7.24) \quad \frac{d}{d\tau} G(AD, C) > l_2e \quad \text{if} \quad |\tau| < \delta.$$

A formula similar to (7.20) holds for  $\mathcal{H}$  also. Here, however, we can use (7.5) and (7.23) to obtain

$$(7.25) \quad \left| \frac{d}{d\tau} \mathcal{H}(AD, C) \right| < l_2[em/M] + \theta Ml_1l_2, \quad |\theta| \leq 1, \\ < 2lem/M.$$

As  $\tau$  traverses the interval  $[-\delta, \delta]$  inequality (7.24) remains valid; so by the theorem of mean value

$$(7.26) \quad G(AD_\delta C) - G(AD_0 C) > \delta l_2e, \\ G(AD_0 C) - G(AD_{-\delta} C) > \delta l_2e.$$

By (7.17) and (7.23)

$$(7.27) \quad |G(AD_0 C) - G(ABC)| < e\delta l_2;$$

so by (7.26) the number  $G(ABC)$  lies between  $G(AD_{-\delta} C)$  and  $G(AD_\delta C)$ . But

$G(AD_\tau C)$  is a continuous function of  $\tau$ ; so there is a value of  $\tau$ , we call it  $\sigma$ , such that

$$(7.28) \quad G(AD_\sigma C) = G(ABC).$$

Using another form of the theorem of mean value, we obtain

$$(7.29) \quad \frac{\mathfrak{K}(AD_\sigma C) - \mathfrak{K}(AD_0 C)}{G(AD_\sigma C) - G(AD_0 C)} = \frac{\frac{d}{d\tau} \mathfrak{K}(AD_\tau C)}{\frac{d}{d\tau} G(AD_\tau C)},$$

the right-hand member being calculated for some  $\tau$  between 0 and  $\sigma$ . By (7.21) and (7.22) the right-hand member of (7.29) has an absolute value less than  $2m/M$ ; so, using (7.29), (7.28), (7.17), and (7.16), we obtain

$$(7.30) \quad \begin{aligned} \mathfrak{K}(AD_\sigma C) - \mathfrak{K}(AD_0 C) &< (2m/M) | G(AD_\sigma C) - G(AD_0 C) | \\ &= (2m/M) | G(ABC) - G(AD_0 C) | \\ &< 2ml_1 l_2 < \mathfrak{K}(ABC) - \mathfrak{K}(AD_0 C). \end{aligned}$$

That is,

$$(7.31) \quad \mathfrak{K}(AD_\sigma C) < \mathfrak{K}(ABC).$$

It follows from (7.28) and (7.31) that

$$(7.32) \quad \begin{aligned} \mathcal{F}(AD_\sigma C) &= \mathfrak{K}(AD_\sigma C) + \lambda(z^*, A) G(AD_\sigma C) \\ &< \mathfrak{K}(ABC) + \lambda(z^*, A) G(ABC) \\ &= \mathcal{F}(ABC). \end{aligned}$$

We have thus shown that if  $ABC$  is in the  $\zeta$ -neighborhood of  $z^*$  and inequalities (7.7) hold, then  $ABC$  fails to minimize  $G(ADC)$  in the class of two-sided polygons  $ADC$  such that  $G(ADC) = G(ABC)$ . For (7.28) and (7.32) show that  $AD_\sigma C$  is in the given class and gives a smaller value to  $\mathcal{F}(C)$ . If we let  $\delta_1$  be the smaller of  $\zeta$  and  $\delta$ , it then serves as the  $\delta$  of the conclusion of the lemma. The proof is therefore complete.

**8. The existence theorem.** We now make another assumption concerning  $\mathcal{F}$  and  $G$ :

(8.1) *For each  $z_0$ , every approach set  $A$  at  $z_0$  consists of the positive multiples of a finite number of unit vectors  $p_1, \dots, p_k$ ; and these can be so ordered that†  $\Omega_H(z_0, p_i, p_i) < 0$  if  $i < j$ .*

We wish to prove the following lemma:

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† As usual,  $H(z, r) = F(z, r) - \lambda(z_0, A)G(z, r)$ .

LEMMA 7. *If  $t_0$  is in  $E$ , then*

$$(8.2) \quad \phi'_0(t_0) = \phi'(t_0)$$

and

$$(8.3) \quad \gamma'_0(t_0) = \gamma'(t_0),$$

where  $\phi, \phi_0, \gamma, \gamma_0$  are the functions defined in §4.

At  $t_0$ , either case (i) or case (ii) of Lemma 5 holds. Suppose case (i). Then there is a sequence  $\{\Pi_m\}$  and a sequence  $t_m \rightarrow t_0$ , each  $t_m$  being contained in an interval  $h_m < t_m < k_m$  on which  $z_m(t)$  is linear, and  $k_m - h_m$  does not tend to zero. We now select a subsequence  $\{\Pi_l\}$  for which  $k_l - h_l$  tends to a limit greater than zero, and further select a subsequence  $\{\Pi_p\}$  of  $\{\Pi_l\}$  such that  $h_p$  and  $k_p$  tend to limits  $h, k$ , respectively. Then  $k > h$ . Also, since  $h_p < t_p < k_p$  and  $t_p \rightarrow t_0$ , it is clear that  $h \leq t_0 \leq k$ . Suppose, to be specific, that  $h \leq t_0 < k$ .

We now choose numbers  $l, m$  such that  $h < l < m < k$ . On the interval  $[l, m]$  all functions  $z_p(t)$  except a finite number are linear. Since  $z_p(t)$  tends to  $z_0(t)$  and is linear on  $[l, m]$ , it is also true that  $z'_p(t)$  tends to  $z'_0(t)$ , and in fact uniformly, on  $[l, m]$ . So

$$\begin{aligned} \phi(m) - \phi(l) &= \lim_{p \rightarrow \infty} (\phi_p(m) - \phi_p(l)) = \lim_{p \rightarrow \infty} \int_l^m F(z_p, z'_p) dt \\ &= \int_l^m F(z_0, z'_0) dt = \phi_0(m) - \phi_0(l). \end{aligned}$$

Since  $\phi$  and  $\phi_0$  are continuous at  $t_0$ , we let  $l \rightarrow t_0$  and obtain

$$\phi(m) - \phi(t_0) = \phi_0(m) - \phi_0(t_0).$$

Dividing by  $m - t_0$  and letting  $m \rightarrow t_0$  gives

$$\phi'(t_0) = \phi'_0(t_0),$$

establishing (8.2). In a like manner we establish (8.3).

Now suppose that case (ii) of Lemma 5 holds. Let  $A$  be an approach set at  $z_0(t_0)$  consisting of the positive multiples of the unit vectors  $p_1, \dots, p_k$  numbered as in (8.1), and let  $\{\Pi_m\}$  be a subsequence of  $\{\Pi_n\}$  such that for every sequence  $t_m \rightarrow t_0$  all accumulation points of the sequence  $z'_m(t_m)$  are in  $A$ . Then if  $\epsilon$  is a positive number, there is a  $\delta_0 > 0$  and an  $m_0$  such that  $z'_m(t)$  is in the  $\epsilon$ -neighborhood of  $A$  if  $|t - t_0| < \delta_0$  and  $m > m_0$ . Otherwise for some  $\epsilon > 0$  we could select a subsequence  $\{\Pi_p\}$  and a sequence  $t_p \rightarrow t_0$  for which  $z'_p(t_p)$  has distance greater than or equal to  $\epsilon$  from  $A$ , so that no accumulation point of this sequence would be in  $A$ .

By Lemma 6, for each pair of numbers  $i$  and  $j > i$  there is a positive num-

ber  $\delta_{ij}$  with the properties specified in Lemma 6. Let  $\delta$  be the least of these. We suppose that the  $\epsilon$  of the preceding paragraph is so small that

$$(8.4) \quad |z'_m(t)/|z'_m(t)| - p_i| < \delta$$

for some  $i$  if  $z'_m(t)$  is in the  $\epsilon$ -neighborhood of  $A$ ; this is possible because of (4.9). Moreover, we suppose  $m_0$  so large and  $\delta_0$  so small that

$$(8.5) \quad |z_m(t) - z_0(t_0)| < \delta \quad \text{if} \quad m > m_0, \quad |t - t_0| < \delta_0.$$

Now denote by  $t_m^1$  the least  $t$  in the interval  $(t_0 - \delta_0, t_0 + \delta_0)$  which defines a vertex of  $\Pi_m$ , and denote by  $T_m$  the greatest such  $t$ . Each side of  $\Pi_m$  between  $z_m(t_m^1)$  and  $z_m(T_m)$  has a direction  $z'_m(t)/|z'_m(t)|$  which differs from one of the  $p_i$  by less than  $\epsilon$ . By Lemma 6 and the minimizing property of each  $\Pi_n$ , no side with direction near  $p_i$  is immediately followed by one with direction near  $p_j$ , ( $i < j$ ). Hence the arc of  $\Pi_m$  between  $z_m(t_m^1)$  and  $z_m(T_m)$  consists of a subarc (possibly empty) along which  $z'_m/|z'_m|$  differs by less than  $\epsilon$  from  $p_1$ , followed by an arc along which  $z'_m/|z'_m|$  differs by less than  $\epsilon$  from  $p_2$ , followed by  $\dots$ , followed by an arc along which  $z'_m/|z'_m|$  differs from  $p_k$  by less than  $\epsilon$ . The values of  $t$  which mark the ends of these arcs we denote by  $t_m^2, t_m^3, \dots, t_m^{k+1} = T_m$ . Thus if  $t_m^j < t < t_m^{j+1}$  and  $t$  does not define a vertex of  $\Pi_m$ , the inequality

$$(8.6) \quad |z'_m(t)/|z'_m(t)| - p_i| < \epsilon$$

holds. We now select a subsequence  $\{\Pi_h\}$  such that  $t_h^j$  converges to a limit  $t^j$  as  $h \rightarrow \infty$ , ( $j=1, \dots, k+1$ ). Then  $t^1 < t_0 < t^{k+1}$ , for otherwise we would be back in case (i) of Lemma 5.

If the interval  $(t^j, t^{j+1})$ , ( $j=1, \dots, k$ ), is not empty, we choose an interval  $(l, m)$  such that  $t^j < l < m < t^{j+1}$ . Then for all large  $h$  the inequality

$$(8.7) \quad |z'_h(t)/|z'_h(t)| - p_j| < \epsilon$$

holds if  $l \leq t \leq m$  and  $z_h(t)$  is not a vertex. From (8.7) and (4.9),

$$(8.8) \quad |z'_h(t) - \mathcal{L}(\Pi_h)p_j| = \mathcal{L}(\Pi_h) \cdot |z'_h(t)/|z'_h(t)| - p_j| < \epsilon L.$$

If  $t$  and  $t+\tau$  are both in  $(l, m)$ , this yields

$$(8.9) \quad \begin{aligned} & |z_h(t+\tau) - z_h(t) - \mathcal{L}(\Pi_h)\tau p_j| \\ &= \left| \int_t^{t+\tau} \{\dot{z}_h(t) - \mathcal{L}(\Pi_h)p_j\} dt \right| \leq \epsilon L |\tau|. \end{aligned}$$

We may without loss of generality suppose that the sequence  $\{\Pi_h\}$  was so chosen that the lengths  $\mathcal{L}(\Pi_h)$  approach a limit  $L_0$ . Then, letting  $h \rightarrow \infty$ , we have

$$(8.10) \quad |z_0(t + \tau) - z_0(t) - L_0 \tau p_i| \leq \epsilon L |\tau|.$$

Now dividing by  $\tau$  and letting  $\tau \rightarrow 0$ , we obtain

$$(8.11) \quad |z'_0(t) - L_0 p_i| \leq \epsilon L,$$

provided only that  $z'_0(t)$  exists. For all large  $h$  we have  $|\mathcal{L}(\Pi_h) - L_0| < \epsilon$ , and (8.8) holds; so if  $l \leq t \leq m$  and  $z'_0(t)$  exists, then

$$(8.12) \quad |z'_h(t) - z'_0(t)| \leq |z'_h(t) - \mathcal{L}(\Pi_h)p_i| + |\mathcal{L}(\Pi_h)p_i - L_0 p_i| \\ + |L_0 p_i - z'_0(t)| < \epsilon(2L + 1).$$

Since  $F$  and  $G$  are continuous at the point  $(z_0(t_0), Lp_i)$ , for every  $\rho > 0$  we can find a neighborhood  $U$  of this point set such that

$$(8.13) \quad |F(z, z') - F(z_1, z'_1)| < \rho$$

and

$$(8.14) \quad |G(z, z') - G(z_1, z'_1)| < \rho$$

if  $(z, z')$  and  $(z_1, z'_1)$  are both in  $U$ . We now suppose that  $\delta$  and  $\epsilon$  are so small and  $h_0$  so large that we have

$$(8.15) \quad (z_h(t), z'_h(t)) \text{ and } (z_0(t), z'_0(t)) \text{ in } U$$

if  $l \leq t \leq m$  and  $h > h_0$  and the derivatives exist (see (8.8) and (8.12)). Then by (8.13), for all  $h > h_0$  we have

$$(8.16) \quad |[\phi_h(m) - \phi_h(l)] - [\phi_0(m) - \phi_0(l)]| = \left| \int_l^m [F(z_h, \dot{z}_h) - F(z_0, \dot{z}_0)] dt \right| \\ \leq \rho(m - l).$$

Letting  $h \rightarrow \infty$ , we get

$$(8.17) \quad |[\phi(m) - \phi(l)] - [\phi_0(m) - \phi_0(l)]| \leq \rho(m - l).$$

Now  $\phi$  is continuous on  $[t_0 - \delta, t_0 + \delta]$ , for  $[z_h^{0'}]$  is bounded away from 0 by (8.6) (if  $\epsilon$  is smaller than  $p_i^0/L$ ), so that  $F(z_h, \dot{z}_h)$  is uniformly bounded. Also,  $\phi_0$  is a.c. by its definition. So we can let  $m \rightarrow t^{i+1}$  and  $l \rightarrow t^i$ , obtaining

$$(8.18) \quad |\{\phi(t^{i+1}) - \phi(t^i)\} - \{\phi_0(t^{i+1}) - \phi_0(t^i)\}| \leq \rho(t^{i+1} - t^i).$$

Although (8.18) was obtained under the assumption  $t^{i+1} > t^i$ , it clearly holds if  $t^{i+1} = t^i$ .

Now we add the relations (8.18) for  $j=0, 1, \dots, k$ . We find

$$(8.19) \quad |\{\phi(t^{k+1}) - \phi(t^1)\} - \{\phi_0(t^{k+1}) - \phi_0(t^1)\}| \leq \rho(t^{k+1} - t^1).$$

We can divide by  $t^{k+1} - t^1$ , since we have seen that  $t^1 < t_0 < t^{k+1}$ . But the num-

ber  $\delta_0$  could be chosen as small as desired, and  $t_0 - \delta \leq t^1 < t^{k+1} \leq t_0 + \delta_0$ . So if we let  $\delta_0$  tend to 0, the difference  $t^{k+1} - t^1$  must also approach 0, and (8.19) yields

$$(8.20) \quad |\phi'(t_0) - \phi'_0(t_0)| \leq \rho.$$

But here  $\rho$  is an arbitrary positive number; so (8.20) implies

$$(8.21) \quad \phi'(t_0) = \phi'_0(t_0).$$

Likewise we can establish (8.3).

This completes the proof of the theorem. For  $\gamma(t)$  and  $\gamma_0(t)$  are a.c., and their derivatives are equal for almost all  $t$ , and  $\gamma(0) = \gamma_0(0) = 0$ ; so  $\gamma(1) = \gamma_0(1)$ . Also,  $\phi(t)$  is monotonic increasing and  $\phi_0(t)$  is a.c., and  $\phi'(t) = \phi'_0(t)$  for almost all  $t$ ; hence

$$\phi(1) \geq \int_0^1 \phi'(t) dt = \int_0^1 \phi'_0(t) dt = \phi_0(1).$$

Thus we have established (4.15), and, as we saw in §4, this implies that  $C_0$  is the minimizing curve sought.

Collecting the hypotheses introduced at various stages of the proof, we obtain the following theorem:

**THEOREM.** *Let the functions  $f(x, y, \dot{y})$  and  $g(x, y, \dot{y})$  be defined and continuous with all their first-order partial derivatives for all  $(x, y, \dot{y})$ . Let  $F(z, z')$  and  $G(z, z')$  be the parametric integrands associated with  $f, g$ , respectively. Assume that hypotheses (2.4), (3.1), (3.2), (3.3), and (8.1) are satisfied. Let  $G(C)$  be quasi-regular normal, and let  $(x_0, y_0)$  and  $(X, Y)$  be two points such that  $x_0 < X$ .*

*Then for every  $l$  the class of a.c. curves  $y = y(x)$ ,  $(x_0 \leq x \leq X)$ , joining  $(x_0, y_0)$  to  $(X, Y)$  and such that  $G(y) = l$  either is empty or contains a minimizing curve for  $\mathcal{F}(C)$ .*

**9. Example.** An example satisfying our conditions is

$$f(x, y, y') = \phi(y)(y'^2 + 1), \quad g(x, y, y') = (1 + y'^2)^{1/2},$$

where  $\phi(y) > 0$  and  $\phi'(y) > 0$ . Transforming to parametric form, we have

$$F(x, y, x', y') = \phi(y) \left( \frac{y'^2}{x'} + x' \right), \quad G(x, y, x', y') = (x'^2 + y'^2)^{1/2}.$$

To determine the approach sets at a fixed point  $(x, y)$ , we recall that, by Lemma 2, if  $(1, q)$  and  $(1, \bar{q})$  form an approach set, then for some number  $\lambda$  these same sets form an approach set for  $f - \lambda g$ . Hence if we consider the graph of  $u = f(x, y, r) - \lambda g(x, y, r)$  as a function of  $r$ , the tangent at  $r = q$  and the

tangent at  $r=\bar{q}$  coincide. (Cf. the geometric interpretation of Lemma 2.) Therefore the graph has at least two flex points between  $q$  and  $\bar{q}$ . But

$$f_{y'y'}(x, y, r) - \lambda g_{y'y'}(x, y, r) = 2\phi(y) - \lambda(1 + r^2)^{-3/2}.$$

This can have at most two zeros, of opposite sign. So  $q$  and  $\bar{q}$  are of opposite sign, and no approach set can contain more than two members of the form  $(1, q_1)$  and  $(1, q_2)$ . We easily see that  $(1, q)$  and  $(1, -q)$  form an approach set. Hence given any  $(x, y, q)$ , we see that the entire approach set at  $(x, y)$  containing  $(1, q)$  consists of the positive multiples of  $(1, q)$  and  $(1, -q)$ .

We readily calculate that, independently of  $\lambda$ ,

$$\Omega_H(x, y; 1, q; 1, -q) = 2q\phi'(y)(q^2 + 1),$$

which is positive if  $q > 0$ . So (8.1) is satisfied. It remains only for us to verify (2.4). In (2.4) we may assume, if we wish, that  $|p_n| = |r_n| = 1$ , since  $F_i(z, r)$  and  $G_i(z, r)$  are positively homogeneous of degree 0 in  $r$ . Then their limits  $p, r$  are also unit vectors. With the assumption, the matrix in (2.4) is, for our example,

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$A_{11} = \phi(\bar{y}_{1,n})\{1 - (r_n^1/r_n^0)^2\} - \phi(y_{1,n})\{1 - (p_n^1/p_n^0)^2\} + \theta_{1,n},$$

$$A_{12} = 2\phi(\bar{y}_{2,n})r_n^1/r_n^0 - 2\phi(y_{2,n})p_n^1/p_n^0 + \theta_{2,n},$$

$$A_{21} = r_n^0 - p_n^0 + \bar{\theta}_{1,n}, \quad A_{22} = r_n^1 - p_n^1 + \bar{\theta}_{2,n}.$$

We must show  $\lim r_n^0 = r^0 > 0$ . Suppose the contrary. Then  $\lim |r_n^1| = 1$ , while  $\lim p_n^1 = p^1 \neq 1$  and  $\lim \bar{\theta}_{2,n} = 0$ . Therefore  $\lim A_{22} \neq 0$ . The last two terms in  $A_{11}$  and the last two in  $A_{12}$  have finite limits. If our matrix has rank less than two, then

$$A_{11}A_{22} - A_{12}A_{21} = 0, \quad n = 1, 2, \dots$$

But  $|r_n^1/r_n^0|$  tends to  $\infty$ , and in the determinant its square occurs with a coefficient which approaches a limit  $\phi(y)(r^1 - p^1)$  which is not zero, while its first power has a bounded coefficient. Thus as  $n \rightarrow \infty$  the absolute value of the determinant will also tend to  $\infty$ , contradicting the assumption that it is zero for all  $n$ . So (2.4) is established, and all the hypotheses of our theorem are satisfied.

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